

Quantum computation in semiconductor quantum dots of electron-spin asymmetric anisotropic exchange

Xiang Hao and Shiqun Zhu*

*School of Physical Science and Technology, Suzhou University,
Suzhou, Jiangsu 215006, People's Republic of China*

The universal quantum computation is obtained when there exists asymmetric anisotropic exchange between electron spins in coupled semiconductor quantum dots. The asymmetric Heisenberg model can be transformed into the isotropic model through the control of two local unitary rotations for the realization of essential quantum gates. The rotations on each qubit are symmetrical and depend on the strength and orientation of asymmetric exchange. The implementation of the axially symmetric local magnetic fields can assist the construction of quantum logic gates in anisotropic coupled quantum dots. This proposal can efficiently use each physical electron spin as a logical qubit in the universal quantum computation.

PACS: 03.67.Lx, 71.70.Ej, 73.21.La

I. INTRODUCTION

The electron or nuclear spins in quantum dots are usually regarded as important candidates for qubits in quantum information because of their long decoherence times [1, 2, 3]. It is shown that any universal quantum computation can be implemented by a series of arbitrary single-qubit rotations and controlled-NOT gate [4]. The recent experiment demonstrates the feasibility of coherent control of single-qubit gates by the short bursts of oscillating magnetic fields [5]. In many spin-based quantum computation schemes, isotropic Heisenberg interactions are dominant since the symmetric Hamiltonian conserves the total spin. However, the general spin-orbit couplings actually exist in quantum dots. The anisotropic interaction between electrons in conduction band is very typical in solids [6, 7, 8, 9]. This coupling can show the Dresselhaus form for the bulk inversion asymmetry [10]. The heterostructure asymmetry can also induce the Rashba coupling [11]. These two couplings can bring out the asymmetric anisotropic Dzyaloshinskii-Moriya (DM) interaction [12, 13]. Unlike the isotropic exchange, the asymmetric spin-orbit coupling can greatly reduce the gate fidelity which cannot be neglected in quantum information processing [14, 15, 16, 17, 18]. To eliminate the impacts of these asymmetric anisotropic couplings, some useful methods of universal quantum computation were used by means of encoding a logical qubit into more physical spins [19, 20, 21, 22]. These encoding schemes will unavoidably waste many resources of physical quantum spins.

In this paper, a method based on the control of symmetric single-qubit rotations in semiconductor quantum dots is proposed. The Hamiltonian of coupled electron spins with asymmetric anisotropic exchange and the transformation method are presented in Sec. II. In Sec. III, the perfect two-qubit gates are realized by the lo-

cal unitary operations in the new Hilbert spin space. A discussion concludes the paper.

II. ASYMMETRIC ANISOTROPIC SPIN COUPLING MODEL

The recent work presented the microscopic description of the interaction between electron spins in two coupled semiconductor quantum dots [9]. Different from the exchange of electrons in conduction band, the asymmetric coupling arises from the Coulomb interaction and from the conduction-valance band mixing. In the twisted spin representation [11], the Hamiltonian of the two asymmetric coupled spins \vec{S}_1 and \vec{S}_2 can be expressed by [9]

$$H = H_S + H_A + H_{DM} \quad (1)$$

where $H_S = J \cos \omega \vec{S}_1 \cdot \vec{S}_2$ represents the symmetric isotropic Heisenberg interaction. The term of $H_A = 2J(\sin \frac{\omega}{2})^2 (\vec{n} \cdot \vec{S}_1)(\vec{n} \cdot \vec{S}_2)$ denotes the symmetric anisotropic interaction. The last term H_{DM} is the main part of asymmetric anisotropic exchange in the Dzyaloshinskii-Moriya form of $H_{DM} = J \sin \omega [\vec{n} \cdot (\vec{S}_1 \times \vec{S}_2)]$. The parameter J is the isotropic Heisenberg exchange constant, $\vec{n} = \vec{b}/|\vec{b}|$ is the orientation of the asymmetric anisotropic exchange \vec{b} . The inherent parameter $\omega = \arctan(|\vec{b}|/J)$. The asymmetric anisotropic exchange \vec{b} depends on the separation distance and the orientation between two coupled quantum dots [9]. It is found that the error of the gate from the asymmetric anisotropic exchange \vec{b} is far beyond the limit of fault tolerant quantum computation 10^{-6} [23, 24, 25]. Therefore, it is necessary to investigate the method to eliminate the effects of the asymmetric exchange.

By diagonalization of the Hamiltonian H , the asymmetric form can be rotated into the isotropic one H_0 in the new spin Hilbert space with

$$H_0 = THT^\dagger \quad (2)$$

*Corresponding author; Electronic address: szhu@suda.edu.cn

where T is the unitary rotation. The new spin space can be expanded by $\{T|00\rangle, T|01\rangle, T|10\rangle, T|11\rangle\}$ where $|i\rangle$, ($i = 0, 1$) is the single-qubit basis of spin operator S^z with the corresponding eigenvalues $\pm\hbar/2$. The definite form of unitary rotation T takes a key part in the following scheme of quantum computation. Without losing

the generality, two typical kinds of asymmetric exchange orientations of $\vec{n}_{xy} = \cos\theta\vec{e}_x + \sin\theta\vec{e}_y$ and $\vec{n}_z = \vec{e}_z$ are considered.

In the case of \vec{n}_{xy} , the four eigenstates of the Hamiltonian H can be written as

$$\begin{aligned} |\varphi_1\rangle &= |\psi^+\rangle \\ |\varphi_2\rangle &= \frac{1}{\sqrt{2}}[(\sin\theta + \cos\theta\cos\frac{\omega}{2})|\phi^-\rangle + i(\cos\theta - \sin\theta\cos\frac{\omega}{2})|\phi^+\rangle - i\sin\frac{\omega}{2}|\psi^-\rangle] \\ |\varphi_3\rangle &= \frac{1}{\sqrt{2}}[(\cos\theta + \sin\theta\cos\frac{\omega}{2})|\phi^+\rangle - i(\sin\theta - \cos\theta\cos\frac{\omega}{2})|\phi^-\rangle + \sin\frac{\omega}{2}|\psi^-\rangle] \\ |\varphi_4\rangle &= -i\cos\theta\sin\frac{\omega}{2}|\phi^-\rangle - \sin\theta\sin\frac{\omega}{2}|\phi^+\rangle + \cos\frac{\omega}{2}|\psi^-\rangle \end{aligned} \quad (3)$$

Here the states $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ and $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ are the four Bell states in the space of $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Meanwhile, the four Bell states are the eigenstates of the isotropic Heisenberg exchange

Hamiltonian H_0 . By means of the representation transformation, the unitary rotation T_{xy} can be given in the space of $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ with

$$T_{xy}(\omega, \theta) = \begin{pmatrix} (\cos\frac{\omega}{4})^2 e^{i(\theta-\frac{\pi}{4})} & \cos\frac{\omega}{4}\sin\frac{\omega}{4}e^{i\frac{\pi}{4}} & -\cos\frac{\omega}{4}\sin\frac{\omega}{4}e^{i\frac{\pi}{4}} & (\sin\frac{\omega}{4})^2 e^{-i(\theta+\frac{\pi}{4})} \\ \cos\frac{\omega}{4}\sin\frac{\omega}{4}e^{i(\theta+\frac{\pi}{2})} & (\cos\frac{\omega}{4})^2 & (\sin\frac{\omega}{4})^2 & \cos\frac{\omega}{4}\sin\frac{\omega}{4}e^{-i(\theta+\frac{\pi}{2})} \\ \cos\frac{\omega}{4}\sin\frac{\omega}{4}e^{i(\theta-\frac{\pi}{2})} & (\sin\frac{\omega}{4})^2 & (\cos\frac{\omega}{4})^2 & \cos\frac{\omega}{4}\sin\frac{\omega}{4}e^{-i(\theta-\frac{\pi}{2})} \\ (\sin\frac{\omega}{4})^2 e^{i(\theta+\frac{\pi}{4})} & \cos\frac{\omega}{4}\sin\frac{\omega}{4}e^{-i\frac{\pi}{4}} & -\cos\frac{\omega}{4}\sin\frac{\omega}{4}e^{-i\frac{\pi}{4}} & (\cos\frac{\omega}{4})^2 e^{-i(\theta-\frac{\pi}{4})} \end{pmatrix} \quad (4)$$

Similarly, the four eigenstates of the Hamiltonian with the orientation \vec{n}_z can be expressed as

$$\begin{aligned} |\varphi_1\rangle &= |\phi^-\rangle \\ |\varphi_2\rangle &= |\phi^+\rangle \\ |\varphi_3\rangle &= \frac{1}{\sqrt{1+(\tan\frac{\omega}{2})^2}}(|\psi^+\rangle + i\tan\frac{\omega}{2}|\psi^-\rangle) \\ |\varphi_4\rangle &= \frac{1}{\sqrt{1+(\tan\frac{\omega}{2})^2}}(|\psi^-\rangle + i\tan\frac{\omega}{2}|\psi^+\rangle) \end{aligned} \quad (5)$$

The definite form of the unitary operation T_z in the space of $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ can also be obtained by the representation transformation

$$T_z(\omega, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\frac{\omega}{2}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\omega}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6)$$

It is clear that the unitary operations T depend only on the orientation of asymmetric anisotropic exchange \vec{n} and the strength ratio of the asymmetric exchange to

the isotropic constant $|\vec{b}|/J$. This means that one kind of inherent quantum-dots structures can determine the corresponding kind of unitary operation form T . After the operation of T , the asymmetric anisotropic term will be eliminated in the Hamiltonian. In the new spin space, the new form of the Hamiltonian takes on the isotropic Heisenberg interaction of $H_0 = J\vec{S}'_1 \cdot \vec{S}'_2$ where the new spin operator can be expressed by $\vec{S}' = T\vec{S}T^\dagger$. The method to apply the unitary operation is one essential aspect in the quantum-dots quantum computation.

III. REALIZATION OF UNIVERSAL QUANTUM LOGICAL GATES

Before further discussions, the major property of the unitary operation T needs to be investigated. It is found that the entanglement of the thermal state $\rho(H)$ keeps the same as that of $\rho(H_0)$. Since the entanglement of the states cannot be varied when the states are transformed by unitary local rotations, the two-qubit operation T can be expressed by the direct product of two unitary local

rotations $T = U_1 \otimes U_2$. Recent experiment implies that arbitrary single-qubit rotations can be implemented in quantum dots [5]. After the operation of T^\dagger , the two-qubit evolution operator can be expressed by

$$\mathcal{U}(t) = \exp\{-i \int_0^t H(t') dt'\} T^\dagger \quad (7)$$

When the evolution operator $\mathcal{U}(t)$ is applied to the initial quantum state $|\Psi(0)\rangle$ for a period of time τ , the intermediate state can be written by $\mathcal{U}(\tau)|\Psi(0)\rangle$. If the unitary rotation T is then employed, the whole quantum state can be obtained by

$$|\Psi(t)\rangle = T \exp\{-i \int_0^t H(t') dt'\} T^\dagger |\Psi(0)\rangle \quad (8)$$

Meanwhile, Eq. (8) can also be expressed as $|\Psi(t)\rangle = \exp\{-i \int_0^t T H(t') T^\dagger dt'\} |\Psi(0)\rangle = \exp\{-i \int_0^t H_0(t') dt'\}$. Thus, when the evolution time satisfies $\int_0^{\tau_s} J dt' = J\tau_s = \pi \pmod{2\pi}$, one can obtain the swap quantum gate \mathcal{U}_{sw} which just exchanges the quantum states between two electron spins [1]. Moreover, with the help of single-qubit gate sequence, the controlled-NOT gate can be constructed by $e^{i\frac{\pi}{2}S_1^z} e^{-i\frac{\pi}{2}S_2^z} \mathcal{U}_{sw}^{1/2} e^{i\pi S_1^z} \mathcal{U}_{sw}^{1/2}$ where the square-root of the swap gate $\mathcal{U}_{sw}^{1/2}$ corresponds to the half evolution time $\frac{\tau_s}{2}$ of the operator \mathcal{U} .

It is important how to apply this unitary transformation T . It is known that the operation T can be decomposed by two local unitary rotations U_1 and U_2 . Arbitrary unitary operation at qubit i can be written as [26]

$$U_i = \Phi_i(\delta) R_i^z(\alpha) R_i^y(\gamma) R_i^z(\beta) \quad (9)$$

where $\Phi_i(\delta) = e^{i\delta I}$ is the phase shift with respect to the angle δ , $R_i^z(\alpha) = e^{i\alpha S_i^z}$ represents the rotation by the angle α about the z -axis and $R_i^y(\beta) = e^{i\beta S_i^y}$ is that by the angle β about the y -axis. It is found that the unitary rotation T can be given by single-qubit rotations as a function of the orientation and the relative strength of the asymmetric exchange. For the two cases of \vec{n}_{xy} and \vec{n}_z , one has

$$\begin{aligned} T_{xy} &= R_1^z(-\frac{3\pi}{4}) R_1^y(\frac{\omega}{2}) R_1^z(\theta + \frac{\pi}{2}) \\ &\otimes R_1^z(\frac{\pi}{4}) R_1^y(\frac{\omega}{2}) R_1^z(\theta - \frac{\pi}{2}) \\ T_z &= R_1^z(-\frac{\omega}{2}) \otimes R_2^z(\frac{\omega}{2}) \end{aligned} \quad (10)$$

This equation demonstrates that the special unitary operation T is accomplished by two local axis-symmetrical rotations. Therefore, if the unitary rotation T is operated in the sequence of THT^\dagger , the asymmetric anisotropic exchange can be rotated into the isotropic Heisenberg model with the same coupling J .

From Eq. (10), It is clear that these local rotations depend critically on the values of the parameters ω and θ . The fidelity of one gate can be expressed by

$$F = \text{Tr}[U^\dagger U_0] / \text{Tr}[U_0^\dagger U_0] \quad (11)$$

where $U_0(\omega_0, \theta_0)$ is the perfect operation without variations and $U(\omega, \theta)$ is the one with small variations of $\Delta\omega = \omega - \omega_0$ and $\Delta\theta = \theta - \theta_0$. The fidelity F of swap gate \mathcal{U}_{sw} and the gate errors ε are calculated and plotted in Fig. 1 in dimensionless unit. The fidelity F is plotted in Fig. 1(a). It is seen that the gate fidelity decreases when the variations $|\Delta\omega|/\omega_0$ and $|\Delta\theta|/\theta_0$ increase. The gate error $\varepsilon = 1 - F$ is shown in Fig. 1(b). The order of the gate errors induced by the asymmetric anisotropic exchange is about $5 \times 10^{-6} \rightarrow 10^{-5}$ which is much higher than the limit of fault tolerant quantum computation 10^{-6} . Even when $|\Delta\theta|/\theta_0$ is varied from 0.1 to 0.01, the errors cannot be reduced and the curves cannot be distinguished. By means of local single-qubit operations $T_{xy}(\omega_0, \theta_0)$ given by Eq. (10), the order of gate errors can be reduced to about 10^{-7} which is much smaller than 10^{-6} . The error is reduced from $5 \times 10^{-7} \rightarrow 10^{-8}$ when $|\Delta\theta|/\theta_0$ is varied from 0.1 to 0.01. This means that the method in this paper is feasible with the small variations of the asymmetric anisotropic exchanges.

If there are external magnetic fields, the general Hamiltonian in the inhomogeneous fields can be expressed as

$$H_B = H + (\vec{B}_1 \cdot \vec{S}_1 + \vec{B}_2 \cdot \vec{S}_2) \quad (12)$$

It is known that the isotropic Heisenberg interaction in the uniform magnetic field can be regarded as one useful model for the universal quantum computation [27]. The Hamiltonian H in Eq. (12) can be rotated into the isotropic one by the operation of THT^\dagger . The condition of what kind of the inhomogeneous fields can be transformed in the same sequence of $T(\vec{B}_1 \cdot \vec{S}_1 + \vec{B}_2 \cdot \vec{S}_2)T^\dagger = B(S_1^z + S_2^z)$ needs to be investigated. It is demonstrated the axially symmetric magnetic fields with the same strength satisfy this condition. For the case of \vec{n}_{xy} , the previous inhomogeneous fields on two qubits need to be applied in the form of

$$\begin{aligned} \vec{B}_1 &= -B \sin \frac{\omega}{2} (\sin \theta \vec{e}_x - \cos \theta \vec{e}_y - \cot \frac{\omega}{2} \vec{e}_z) \\ \vec{B}_2 &= B \sin \frac{\omega}{2} (\sin \theta \vec{e}_x - \cos \theta \vec{e}_y + \cot \frac{\omega}{2} \vec{e}_z) \end{aligned} \quad (13)$$

Similarly, when the orientation of the asymmetric exchange $\vec{n}_z = \vec{e}_z$, the magnetic fields on two qubits satisfy $\vec{B}_1 = \vec{B}_2 = B\vec{e}_z$. Therefore, after the certain period of time $\frac{\tau_s}{2}$, the phase-shifted swap action can be constructed

$$\begin{aligned} \mathcal{U}_{psw} &(a_1|0\rangle + b_1|1\rangle)_i \otimes (a_2|0\rangle + b_2|1\rangle)_{i+1} \\ &= (a_2 e^{\frac{1}{2}g\mu_B B\tau_s} |0\rangle + b_2|1\rangle)_i \otimes (a_1|0\rangle + b_1|1\rangle)_{i+1} \end{aligned} \quad (14)$$

The controlled-NOT gate can be realized by the phase-shifted swap gate and single-qubit rotations [27].

IV. DISCUSSION

It is found that the asymmetric interactions can be transformed into the isotropic Heisenberg model after the

operation of local symmetric rotations $U_1 \otimes U_2$. The local operations on each qubit are determined by the orientation and the relative strength of asymmetric exchange to the isotropic constant. The gate fidelity is decreased when the small variations of the asymmetric anisotropy increase. When the asymmetric anisotropy is eliminated by the transformation, the order of gate errors is reduced to about 10^{-8} which is much smaller than the limit of fault tolerant quantum computation. Some special inhomogeneous magnetic fields can also contribute to the realization of quantum logic gates. The proposal can uti-

lize each physical electron spin as a logical qubit without encoding many spins.

Acknowledgement

It is a pleasure to thank Yinsheng Ling, Jianxing Fang and Qing Jiang for their many fruitful discussions about the topic. The financial supports from the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20050285002) and the National Natural Science Foundation of China (Grant No. 10774108) are gratefully acknowledged.

-
- [1] D. Loss and D. P. DiVincenzo, Phys. Rev. **A57**, 120(1998).
 - [2] R. Vrijen, E. Yablonovitch, K. Wang, H. W. Jiang, A. Balandin, V. Roychowdhury, T. Mor, and D. P. DiVincenzo, Phys. Rev. **A62**, 012306(2000).
 - [3] B. E. Kane, Nature **393**, 133(1998).
 - [4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*(Cambridge University Press, Cambridge, UK, 2000).
 - [5] F. H. L. Koppens, C. Buizert, K. J. Tielrooij, I. T. Vink, K. C. Nowack, T. Meunier, L. P. Kouwenhoven, and L. M. K. Vandersypen, Nature **442**, 766(2006).
 - [6] K. V. Kavokin, Phys. Rev. **B64**, 075305(2001).
 - [7] L. P. Gor'kov and P. L. Krotkov, Phys. Rev. **B68**, 155206(2003).
 - [8] K. V. Kavokin, Phys. Rev. **B69**, 075302(2004).
 - [9] S. C. Bădescu, Y. B. Lyanda-Geller, and T. L. Reinecke, Phys. Rev. **B72**, 161304(R) (2005).
 - [10] G. Dresselhaus, Phys. Rev. **100**, 580(1955).
 - [11] L. S. Levitov and E. I. Rashba, Phys. Rev. **B67**, 115324(2003).
 - [12] I. Dzyaloshinskii, Phys. Chem. Solids **4**, 241(1958).
 - [13] T. Moriya, Phys. Rev. **120**, 91(1960).
 - [14] N. E. Bonesteel, D. Stepanenko, and D. P. DiVincenzo, Phys. Rev. Lett. **87**, 207901(2001).
 - [15] G. Burkard and D. Loss, Phys. Rev. Lett. **88**, 047903(2002).
 - [16] L. A. Wu and D. A. Lidar, Phys. Rev. **A66**, 062314(2002).
 - [17] D. Stepanenko, N. E. Bonesteel, D. P. DiVincenzo, G. Burkard, and D. Loss, Phys. Rev. **B68**, 115306(2003).
 - [18] D. Stepanenko and N. E. Bonesteel, Phys. Rev. Lett. **93**, 140501(2004).
 - [19] D. P. DiVincenzo et. al. , Nature **408**, 339(2000).
 - [20] S. C. Benjamin, Phys. Rev. **A64**, 054303(2001).
 - [21] D. A. Lidar and L. A. Wu, Phys. Rev. Lett. **88**, 017905(2001).
 - [22] J. Levy, Phys. Rev. Lett. **89**, 147902(2002).
 - [23] P. W. Shor, Phys. Rev. **A52**, R2493(1995).
 - [24] D. Gottesman, Phys. Rev. **A57**, 127(1998).
 - [25] J. Kempe, D. Bacon, D. A. Lidar, and K. B. Whaley, Phys. Rev. **A63**, 042307(2001).
 - [26] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter Phys. Rev. **A52**, 3457(1995).
 - [27] X. Hu, R. de Sousa, and S. Das Sarma, Phys. Rev. Lett. **86**, 918(2001).

Fig. 1 (a). The fidelity of swap gate is plotted with small variations of the asymmetric anisotropy when $\tan \omega_0 = 5 \times 10^{-3}$ [9] and $\theta_0 = 5\pi/6$ radius; (b). The gate error $\log_{10} \varepsilon$ is plotted when $\Delta\theta/\theta_0 = 0.1, 0.01$. The dotted line denotes the limit of fault tolerate quantum computation. The two solid lines above the dotted line are the gate errors induced by \vec{b} . However, they are not distinguishable. The two lines below the dotted line are the errors induced by performing unitary local rotations $T_{xy}(\omega_0, \theta_0)$ with $\Delta\theta/\theta_0 = 0.1, 0.01$ (from upper to lower).

